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Some asymmetric Stokes-flow problems

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Abstract. Solutions are given to a number of asymmetric Stokes-flow problems which involve the slow motion of a fluid in the presence of a rigid plane containing a circular hole. The particular instances of motion generated by a linear shear flow along the plane and by a Stokeslet, placed on the axis of symmetry of the hole and oriented perpendicular to this axis, are examined in detail. The paper concludes with a solution to the shear-flow problem when the circular hole is replaced by one of elliptical geometry.

1. Introduction

In the theory of filtration flows an important problem concerns the determination of the force and torque experienced by a sedimenting particle in the proximity of the pore. A simple model of the pore neglects its length and replaces the fluid-pore geometry by a thin rigid screen pierced by a circular hole and surrounded by incompressible viscous fluid. When the sedimenting particle is small compared with the pore radius and the Reynolds number of the flow is sufficiently small to permit the Stokes linearization of the equations of fluid motion, a translating or rotating particle can be modelled by means of a Stokeslet or rotlet, dipoles or higher-order singularities being unnecessary for the computation of lowest-order wall-effects on the drag and couple experienced by the particle. Thus, in the case of a zero-length pore, Davis et al. [1] (see also Hasimoto [2]) have solved the axisymmetric Stokeslet and rotlet problems in which the singularity lies on the axis of symmetry of the screen. Further, they use the results to compute approximations to the force and couple experienced by a small body, employing formulae due to Brenner [3]. The more complicated problem of a semi-infinite length pore communicating with a half-space chamber of fluid has been treated by Shail and Packham [4], and an investigation of the axisymmetric finite-length pore configuration is currently being undertaken.

All the afore-mentioned research relates to axisymmetric flows, and the purpose of this paper is to treat some asymmetric filtration flows, with the pore again modelled by a hole in a plane rigid wall. Two particular problems are solved, the first being a singularity-free motion in which a linear shear flow exists along the plane wall in one half-space of fluid, the motion being communicated via the pore to the liquid in the other half-space. A solution to this problem, and the two-dimensional situation in which the circular hole is replaced by a slit, has recently been given by Smith [5], but our approach is somewhat different from his. In the second configuration the Stokeslet in [1, 4] is replaced by one again on the axis of symmetry, but now oriented perpendicular to it. This solution is then used in conjunction with [3] to derive an approximation to the force on an arbitrary body when incident with the axis of symmetry as it sediments parallel to the membrane.

In Section 2 we derive a suitable representation of the quasi-steady Stokes velocity and pressure fields in terms of potential functions, and apply them to the auxiliary problem of a Stokeslet placed parallel to and in front of a rigid flat wall. In Section 3 both the shear-flow and Stokeslet problem are solved, and the force calculation is given. Finally, in Section 4 the solution is found to a further shear-flow configuration, in which the circular pore is replaced by an elliptical geometry.

2. Basic equations and solutions

The continuity and linearized Navier–Stokes equations governing the steady creeping flow of an imcompressible fluid in a singularity-free region are

$$\operatorname{div} \mathbf{v} = 0, \tag{1}$$

and

$$\mu \operatorname{curl} \operatorname{curl} \mathbf{v} = -\nabla p, \tag{2}$$

where $\mathbf{v}$ is the velocity field, μ the coefficient of viscosity and p the pressure of the fluid. Let (ϱ, ϕ, z) denote the cylindrical polar coordinates of the point with position vector $\mathbf{r}$, and denote by $\boldsymbol{\varrho}$, $\boldsymbol{\phi}$, $\mathbf{z}$ unit vectors in the directions of ϱ-, ϕ- and z-increasing, respectively. Three solutions $(\mathbf{v}_i, p_i)$, $i = 1, 2, 3$, of (1) and (2), used by the authors in earlier work [4, 6, 7], are given by

$$\mathbf{v}_1 = z\nabla\left(\frac{\partial X}{\partial z}\right) - \frac{\partial X}{\partial z}\mathbf{z} + \nabla X, \quad p_1 = 2\mu\frac{\partial^2 X}{\partial z^2}, \tag{3}$$

$$\mathbf{v}_2 = z\nabla\Psi - \Psi\mathbf{z}, \quad p_2 = 2\mu\frac{\partial \Psi}{\partial z}, \tag{4}$$

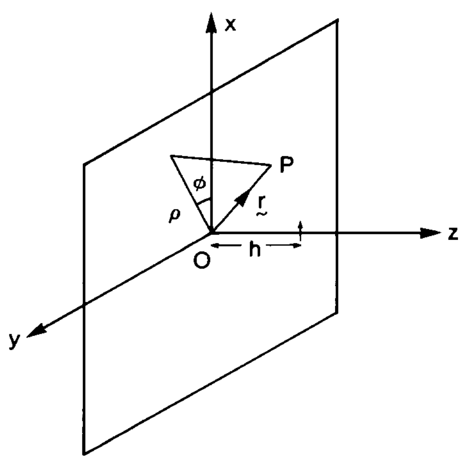

Fig. 1. A Stokeslet placed parallel to the x-axis in front of the rigid plate $z = 0$.

and

$$\mathbf{v}_3 = \operatorname{curl}(\Theta z), \quad p_3 = 0, \tag{5}$$

where X, Ψ, Θ are harmonic functions of (ϱ, ϕ, z). The significant features of (3) and (4) are that $\mathbf{v}_1 \cdot \mathbf{z} = 0$ and $\mathbf{v}_2 \cdot \boldsymbol{\varrho} = 0$ on $z = 0$, whilst (5) represents a swirling flow with zero component of velocity in the z-direction.

Suppose now that fluid fills the half-space $z > 0$ with $z = 0$ occupied by an infinite rigid plate. Fluid motion in $z > 0$ is generated by a Stokeslet (point force) of unit strength placed on the z-axis at a distance h from the bounding plane, and oriented parallel to the positive Cartesian x-axis (see Fig. 1). In an infinite fluid the Stokeslet produces a velocity field $\mathbf{v}_\infty(\mathbf{r}, h)$ with cylindrical components

$$\begin{aligned} u_\infty &= \left\{\frac{2}{R_1} - \frac{(z-h)^2}{R_1^3}\right\} \cos\phi, \\ v_\infty &= -\frac{\sin\phi}{R_1}, \\ w_\infty &= \frac{\varrho(z-h)}{R_1^3} \cos\phi, \end{aligned} \tag{6}$$

where $R_1 = \{\varrho^2 + (z-h)^2\}^{1/2}$, the corresponding pressure field $p_\infty(\mathbf{r}, h)$ being

$$p_\infty = \frac{2\mu\varrho}{R_1^3} \cos\phi. \tag{7}$$

To solve the problem with the barrier $z = 0$ present we use (3), (5) and an image Stokeslet to express the velocity and pressure fields as

$$\mathbf{v} = \mathbf{v}_\infty(\mathbf{r}, h) + \mathbf{v}_\infty(\mathbf{r}, -h) + z\nabla\left(\frac{\partial X_0}{\partial z}\right) - \frac{\partial X_0}{\partial z}\mathbf{z} + \nabla X_0 + \operatorname{curl}(\Theta_0 \mathbf{z}), \tag{8}$$

$$p = p_\infty(\mathbf{r}, h) + p_\infty(\mathbf{r}, -h) + 2\mu \frac{\partial^2 X_0}{\partial z^2}. \tag{9}$$

Writing the cylindrical polar components of velocity (u, v, w) as

$$u = u_1(\varrho, z)\cos\phi, \quad v = v_1(\varrho, z)\sin\phi, \quad w = w_1(\varrho, z)\cos\phi,$$

and the pressure p as $P_1(\varrho, z)\cos\phi$, then from (8) and (9),

$$u_1 = 2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \left\{\frac{(z-h)^2}{R_1^3} + \frac{(z+h)^2}{R_2^3}\right\} + z\frac{\partial^2 \chi_0}{\partial\varrho\partial z} + \frac{\partial\chi_0}{\partial\varrho} + \frac{\theta_0}{\varrho}, \tag{10}$$

$$v_1 = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{z}{\varrho}\frac{\partial \chi_0}{\partial z} - \frac{\chi_0}{\varrho} - \frac{\partial \theta_0}{\partial \varrho}, \tag{11}$$

$$w_1 = \varrho\left\{\frac{(z-h)}{R_1^3} + \frac{(z+h)}{R_2^3}\right\} + \frac{z\partial^2 \chi_0}{\partial z^2}, \tag{12}$$

$$P_1 = 2\mu\varrho\left\{\frac{1}{R_1^3} + \frac{1}{R_2^3}\right\} + 2\mu\frac{\partial^2 \chi_0}{\partial z^2}, \tag{13}$$

where $R_2 = \{\varrho^2 + (z+h)^2\}^{1/2}$. In (10) to (13) χ_0 and θ_0 are related to X_0 and Θ_0 by

$$X_0(\varrho, \phi, z) = \chi_0(\varrho, z)\cos\phi, \quad \Theta_0(\varrho, z, \phi) = \theta_0(\varrho, z)\sin\phi,$$

and θ_0, χ_0 both satisfy the equation

$$\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial f}{\partial\varrho}\right) + \frac{\partial^2 f}{\partial z} - \frac{f}{\varrho^2} = 0. \tag{14}$$

The functions χ_0, θ_0 in (11)–(13) are required to be bounded as $\varrho^2 + z^2 \to \infty$, and must be determined so that the no-slip conditions are satisfied on the boundary $z = 0$. Clearly (12) ensures that $w_1 = 0$ on $z = 0$, whilst $u_1 + v_1 = 0$ on $z = 0$ requires that

$$\left(\frac{\partial}{\partial\varrho} - \frac{1}{\varrho}\right)(\chi_0 - \theta_0) = 2\left(\frac{h^2}{R^3} - \frac{1}{R}\right), \tag{15}$$

where $R = (\varrho^2 + h^2)^{1/2}$. Integrating (15) and rejecting the complementary solution $\chi_0 - \theta_0 = \text{const}.\varrho$ which is unbounded at infinity shows that

$$\chi_0 - \theta_0 = \frac{2\varrho}{R} \quad \text{on} \quad z = 0, \quad 0 \leqslant \varrho < \infty. \tag{16}$$

Similarly the requirement that $u_1 - v_1 = 0$ on $z = 0$ leads to the equation

$$\chi_0 + \theta_0 = \frac{C_1}{\varrho} - \frac{2(4h^2 + 3\varrho^2)}{\varrho R} \quad \text{on} \quad z = 0, \quad 0 \leqslant \varrho < \infty, \tag{17}$$

where C_1 is a constant of integration. In order that $\chi_0 + \theta_0$, and hence u_1, v_1, are bounded on $z = 0$ as $\varrho \to 0$, we must choose $C_1 = 8h$, whence (16) and (17) lead to the conditions

$$\left.\begin{aligned} \chi_0(\varrho, 0) &= \frac{4h}{\varrho} - \frac{2\varrho}{R} - \frac{4h^2}{\varrho R}, \\ \theta_0(\varrho, 0) &= \frac{4}{\varrho}(h - R). \end{aligned}\right\} \quad 0 \leqslant \varrho < \infty. \tag{18}$$

Using these boundary values, the determination of χ_0 and θ_0 can now be completed in an elementary manner using solutions of (14) in the form of Hankel transforms of order one, producing

$$\chi_0(\varrho, z) = \frac{2(z+2h)}{\varrho} - \frac{2h(h+z)}{\varrho R_2} - \frac{2R_2}{\varrho}, \tag{19}$$

$$\theta_0(\varrho, z) = \frac{4}{\varrho}(h + z - R_2). \tag{20}$$

All velocity and stress components etc. can now be computed explicity.

3. The pore-flow problems

We now turn to the problems described in the introduction in which a pore is modelled by a circular hole in a thin rigid plate occupying $z = 0$ (see Fig. 2). The plate is surrounded by viscous incompressible fluid and we label by I and II physical quantities pertaining to the regions $z > 0$ and $z < 0$, respectively.

Suppose that in $z > 0$ there exists a flow (generated, for example, by singularities but with no solid bodies present in $z > 0$) which, when $z = 0$ is a rigid barrier, has velocity and pressure fields $\mathbf{v}_0^{\mathrm{I}}$, p_0^{I}, where in cylindrical polar components

$$\begin{aligned} \mathbf{v}_0^{\mathrm{I}} &= (u_0(\varrho, z)\cos\phi,\ v_0(\varrho, z)\sin\phi,\ w_0(\varrho, z)\cos\phi), \\ p_0^{\mathrm{I}} &= p_0(\varrho, z)\cos\phi. \end{aligned} \tag{21}$$

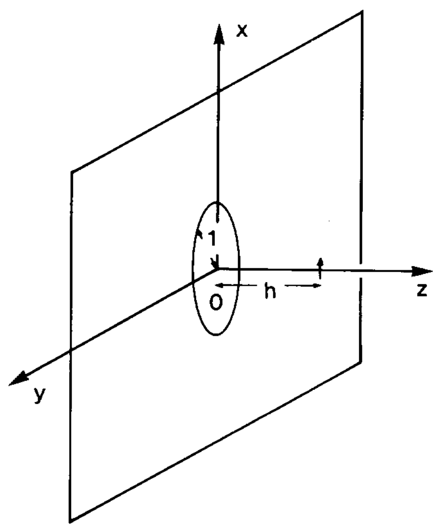

Fig. 2. A Stokeslet placed parallel to the x-axis in front of a rigid plate containing a hole of unit radius.

When the rigid plane is replaced by a plane with the circular region $0 \leqslant \varrho \leqslant 1, 0 \leqslant \phi < 2\pi$, $z = 0$ removed, thus allowing communication with an infinite half-space of similar fluid occupying $z < 0$, we represent the velocity fields by

$$\begin{aligned} \mathbf{v}^{\mathrm{I}}(\varrho, \phi, z) &= \mathbf{v}_0^{\mathrm{I}}(\varrho, \phi, z) + \mathbf{V}(\varrho, \phi, z), \quad z > 0, \\ \mathbf{v}^{\mathrm{II}}(\varrho, \phi, z) &= \mathbf{V}(\varrho, \phi, z), \quad z < 0, \end{aligned} \tag{22}$$

where $\mathbf{V} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ (see (3)–(5)). Since $\mathbf{v}_0^{\mathrm{I}}$ vanishes on $z = 0$, (22) ensures that the velocity is continuous across $z = 0$ if each of the harmonics X, Θ, Ψ is twice continuously differentiable across $z = 0$ (except possibly at the rim of the hole) and even in z. The associated pressure fields are

$$\begin{aligned} p^{\mathrm{I}}(\varrho, \phi, z) &= p_0^{\mathrm{I}} + p_1 + p_2, \quad z > 0, \\ p^{\mathrm{II}}(\varrho, \phi, z) &= p_1 + p_2, \quad z < 0. \end{aligned} \tag{23}$$

Analogous to (10) through (13) we have in $z > 0$ the azimuthal-independent velocity components

$$\begin{aligned} u_1^{\mathrm{I}}(\varrho, z) &= u_0(\varrho, z) + z\frac{\partial^2\chi}{\partial\varrho\partial z} + \frac{\partial\chi}{\partial\varrho} + \frac{\theta}{\varrho} + z\frac{\partial\psi}{\partial\varrho}, \\ v_1^{\mathrm{I}}(\varrho, z) &= v_0(\varrho, z) - \frac{z}{\varrho}\frac{\partial\chi}{\partial z} - \frac{\chi}{\varrho} - \frac{\partial\theta}{\partial\varrho} - \frac{z}{\varrho}\psi, \\ w_1^{\mathrm{I}}(\varrho, z) &= w_0(\varrho, z) + z\frac{\partial^2\chi}{\partial z^2} + z\frac{\partial\psi}{\partial z} - \psi, \end{aligned} \tag{24}$$

where $\mathrm{X}(\varrho, \phi, z) = \chi(\varrho, z)\cos\phi$ etc., whereas in $z < 0$, u_1^{II} etc. follow from u_1^{I} by omitting the contributions with zero subscript. In a similar manner the corresponding azimuthal-independent stress components in $z > 0$ are found as

$$\begin{aligned} \sigma_{\varrho z}^{\mathrm{I}}(\varrho, z) &= \sigma_{\varrho z}^{0}(\varrho, z) + \mu\left(2z\frac{\partial^3\chi}{\partial\varrho\partial z^2} + 2\frac{\partial^2\chi}{\partial\varrho\partial z} + \frac{1}{\varrho}\frac{\partial\theta}{\partial z} + 2z\frac{\partial^2\psi}{\partial\varrho\partial z}\right), \\ \sigma_{\phi z}^{\mathrm{I}}(\varrho, z) &= \sigma_{\phi z}^{0}(\varrho, z) + \mu\left(-\frac{2z}{\varrho}\frac{\partial^2\chi}{\partial z^2} - \frac{2}{\varrho}\frac{\partial\chi}{\partial z} - \frac{\partial^2\theta}{\partial\varrho\partial z} - \frac{2z}{\varrho}\frac{\partial\psi}{\partial z}\right), \\ \sigma_{zz}^{\mathrm{I}}(\varrho, z) &= \sigma_{zz}^{0}(\varrho, z) + 2\mu\left(z\frac{\partial^3\chi}{\partial z^3} + z\frac{\partial^2\psi}{\partial z^2} - \frac{\partial\psi}{\partial z}\right), \end{aligned} \tag{25}$$

where in an obvious notation $\sigma_{\varrho z}^{0}$ etc. are the stresses arising from the basic flow in $z > 0$. (The azimuthal dependencies of the two shears and the normal stress are $\cos\phi$, $\sin\phi$ and $\cos\phi$, respectively.) Again, in $z < 0$ $\sigma_{zz}^{\mathrm{II}}$ etc. is obtained from σ_{zz}^{I} by omitting the zero-superscript basic-flow stresses. We now apply these representations to the shear-flow and Stokeslet problems.

(a) Shear flow past the pore

Suppose that a shear flow parallel to the Cartesian x-axis exists in $z > 0$, the velocity gradient being U; then in (21),

$$\mathbf{v}_0^{\mathrm{I}} = (Uz\cos\phi, -Uz\sin\phi, 0), \quad p_0^{\mathrm{I}} = 0, \tag{26}$$

and the stress components $\sigma^0_{\varrho z}$ etc. are

$$\sigma^0_{\varrho z} = \mu U, \quad \sigma^0_{\phi z} = -\mu U, \quad \sigma^0_{zz} = 0. \tag{27}$$

Following the arguments leading to (16) and (17), $(24)_{1,2}$ and the no-slip conditions on $z = 0+$, $\varrho > 1$, require that

$$\chi(\varrho, 0) = \theta(\varrho, 0) = \frac{C_1}{2\varrho}, \quad \varrho > 1, \tag{28}$$

where C_1 is a constant of integration to be determined. Similarly, from $(24)_3$, the no-slip condition on the z-component of velocity gives

$$\psi(\varrho, 0) = 0, \quad \varrho > 1. \tag{29}$$

Across the mouth of the pore, the components of stress must be continuous. Forming $\sigma_{\varrho z} + \sigma_{\phi z}$ in the regions I and II and taking account of (27) and the even parity of χ and θ in z, the resulting conditions are found as

$$\frac{\partial}{\partial\varrho}\left\{\frac{1}{\varrho}\left(2\frac{\partial\chi}{\partial z}(\varrho, 0+) - \frac{\partial\theta}{\partial z}(\varrho, 0+)\right\} = 0, \tag{30}$$

and

$$\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left\{\varrho\left(2\frac{\partial\chi}{\partial z}(\varrho, 0+) + \frac{\partial\theta}{\partial z}(\varrho, 0+)\right\} = -U, \tag{31}$$

both for $0 \leqslant \varrho < 1$. Integrating with respect to ϱ and rejecting a complementary solution which is unbounded as $\varrho \to 0$ give

$$2\frac{\partial\chi}{\partial z} - \frac{\partial\theta}{\partial z} = C_2\varrho \quad \text{on} \quad z = 0+, \quad 0 \leqslant \varrho < 1,$$

and

$$2\frac{\partial\chi}{\partial z} + \frac{\partial\theta}{\partial z} = -\tfrac{1}{2}U\varrho \quad \text{on} \quad z = 0+, \quad 0 \leqslant \varrho < 1,$$

i.e., on $z = 0+, 0 \leqslant \varrho < 1$,

$$\frac{\partial \chi}{\partial z} = \tfrac{1}{4}(C_2 - \tfrac{1}{2}U)\varrho, \qquad \frac{\partial \theta}{\partial z} = -\tfrac{1}{2}(C_2 + \tfrac{1}{2}U)\varrho, \tag{32}$$

where C_2 is a presently unknown constant of integration. The final stress continuity condition on σ_{zz} is easily found to yield

$$\frac{\partial \psi}{\partial z}(\varrho, 0+) = 0, \quad 0 \leqslant \varrho < 1, \tag{33}$$

and it immediately follows from (29) and (33) that $\psi(\varrho, z) \equiv 0$.

The mixed boundary-value problems for χ, θ in $z \geqslant 0$ are specified by (28) and (32) and both have the same structure. Thus we write $\chi = \chi_1 + \chi_2$, where, on $z = 0+$, χ_1 satisfies

$$\frac{\partial \chi_1}{\partial z} = 0, \quad 0 \leqslant \varrho < 1, \quad \chi_1 = \frac{C_1}{2\varrho}, \quad \varrho > 1, \tag{34}$$

and for χ_2,

$$\frac{\partial \chi_2}{\partial z} = \tfrac{1}{4}(C_2 - \tfrac{1}{2}U)\varrho, \quad 0 \leqslant \varrho < 1, \quad \chi_2 = 0, \quad \varrho > 1. \tag{35}$$

Shail [8] has given the following expression for χ_1:

$$\chi_1(\varrho, z) = \tfrac{1}{2}C_1[1 - \operatorname{Im}\{\varrho^2 + (z+i)^2\}^{1/2}] = \frac{C_1}{2(1+\xi^2)^{1/2}}\left(\frac{1-\eta}{1+\eta}\right)^{1/2}, \tag{36}$$

where (ξ, η) are oblate spheroidal coordinates with $\xi + i\eta = \{\varrho^2 + (z+i)^2\}^{1/2}$. For χ_2 we use a contour-integral type representation (Collins [9]) of the solution of (14) satisfying $(35)_2$, namely

$$\chi_2(\varrho, z) = \frac{1}{2\varrho}\int_{-1}^{1} \frac{(z+it)g(t)}{\{\varrho^2 + (z+it)^2\}^{1/2}}\,\mathrm{d}t, \tag{37}$$

where $g(t)$ is an even continuous function of t which, in order to ensure boundedness of χ_2 on the axis $\varrho = 0$, must satisfy the subsidiary condition

$$\int_0^1 g(t)\,\mathrm{d}t = 0. \tag{38}$$

(In $(36)_1$ and (37) the radical is interpreted as

$$\{\varrho^2 + (z + it)^2\}^{1/2} = \begin{cases} (\varrho^2 - t^2)^{1/2} & \text{if } \varrho > t, \\ i(t^2 - \varrho^2)^{1/2} & \text{if } \varrho < t. \end{cases}$$

as $z \to 0+$.)

Applying $(35)_1$ produces the integral equation

$$-\frac{\partial}{\partial \varrho}\int_0^{\varrho} \frac{g(t)}{(\varrho^2 - t^2)^{1/2}}\, \mathrm{d}t = \tfrac{1}{4}\varrho(C_2 - \tfrac{1}{2}U), \quad 0 \leqslant \varrho < 1, \tag{39}$$

with solution

$$g(t) = -\frac{1}{2\pi}(C_2 - \tfrac{1}{2}U)t^2 + C_3, \quad 0 \leqslant t \leqslant 1,$$

the further constant C_3 being evaluated in terms of C_2 using (38) to provide

$$g(t) = \frac{1}{2\pi}(C_2 - \tfrac{1}{2}U)(\tfrac{1}{3} - t^2), \quad 0 \leqslant t \leqslant 1. \tag{40}$$

Turning next to $\theta(\varrho, z)$ which satisfies the mixed conditions (28), $(32)_2$, we represent it as

$$\theta(\varrho, z) = \chi_1(\varrho, z) + \frac{1}{2\varrho}\int_{-1}^{1} \frac{(z + it)h(t)}{\{\varrho^2 + (z + it)^2\}^{1/2}}\, \mathrm{d}t, \tag{41}$$

where $h(t) = h(-t)$ and

$$\int_0^1 h(t)\, \mathrm{d}t = 0.$$

Repeating the arguments of the previous paragraph gives $h(t)$ as

$$h(t) = -\frac{1}{\pi}(C_2 + \tfrac{1}{2}U)(\tfrac{1}{3} - t^2), \quad 0 \leqslant t \leqslant 1, \tag{42}$$

and it remains to determine the constants C_1, C_2. These are found by requiring that the shear components $\sigma^{\mathrm{I}}_{\varrho z}$, $\sigma^{\mathrm{I}}_{\phi z}$ are no more singular than $(\varrho^2 - 1)^{-1/2}$ as $\varrho \to 1+$ on $z = 0+$, conditions which from $(25)_{1,2}$ require that $\partial^2\chi/\partial\varrho\partial z$ and $\partial^2\theta/\partial\varrho\partial z$ are no more singular than $(\varrho^2 - 1)^{-1/2}$ as $\varrho \to 1+$ on $z = 0+$. From (36), (37) and (36), (41) detailed calculation shows that the appropriate singularity strengths imply that

$$\tfrac{1}{2}C_1 - g(1) = 0, \quad \tfrac{1}{2}C_1 - h(1) = 0. \tag{43}$$

Using (40) and (42) in (43), the constants C_1, C_2 are

$$C_1 = \frac{4U}{9\pi}, \quad C_2 = -\tfrac{1}{6}U,$$

whence

$$g(t) = h(t) = \frac{U}{3\pi}(t^2 - \tfrac{1}{3}), \tag{44}$$

and $\theta(\varrho, z) \equiv \chi(\varrho, z)$.

The simple quadratic expression for $g(t)$ makes an explicit determination of $\chi(\varrho, z)$ a straightforward matter for the shear-flow problem. Substituting (44) in (37) and effecting the integration gives

$$\chi(\varrho, z) = \frac{2U}{9\pi\varrho} + \frac{2U}{9\pi\varrho}\,\mathrm{Im}\,\{\varrho^2 + (z + it)^2\}^{1/2} - \frac{Uz}{3\pi\varrho}\,\mathrm{Im}\,[(z + i)\{\varrho^2 + (z + it)^2\}^{1/2}$$
$$+ \varrho^2 \log [z + i + \{\varrho^2 + (z + it)^2\}^{1/2}]]. \tag{45}$$

When expressed in terms of oblate spheroidal coordinates (ξ, η) by means of the transformation

$$\varrho = \{(1 - \eta^2)(1 + \xi^2)\}^{1/2}, \quad z = \xi\eta,$$

with $\xi + i\eta = \{\varrho^2 + (z + i)^2\}^{1/2}$, $0 \leqslant \xi < \infty$, $-1 \leqslant \eta \leqslant 1$, (45) assumes, for $z \geqslant 0$ (i.e., $0 \leqslant \eta \leqslant 1$), the compact form

$$\chi(\xi, \eta) = \frac{U}{9\pi}\left\{\frac{1 - \eta}{(1 + \eta)(1 + \xi^2)}\right\}^{1/2} \{\eta(\eta + 1)(2 + 3\xi^2) + 2\}$$
$$- \frac{U}{3\pi}\xi\eta\{(1 + \xi^2)(1 - \eta^2)\}^{1/2} \tan^{-1}\left(\frac{1}{\xi}\right). \tag{46}$$

Indeed (46) can be deduced by working from the outset in terms of oblate spheroidal harmonics; however, the method of solution we have given carries over immediately to the Stokeslet problem which is not capable of a simple solution in spheroidal coordinates.

(b) The asymmetric Stokeslet problem

Consider next the configuration in which the basic flow in $z > 0$ is that due to a Stokeslet of unit strength, placed on the z-axis at $z = h$ and oriented parallel to the positive x-axis as in Fig. 2. The velocity field $\mathbf{v}_0^{\mathrm{I}}$ is that provided by (8) with the χ_0- and θ_0-parts of X_0 and Θ_0 given by (19), (20). The associated stresses can be evaluated everywhere in $z > 0$ but of particular interest are the values of $\sigma^0_{\varrho z}(\varrho, 0+)$, $\sigma^0_{\phi z}(\varrho, 0+)$ and $\sigma^0_{zz}(\varrho, 0+)$, $0 \leqslant \varrho < \infty$, which are found as

$$\begin{aligned} \sigma^0_{\varrho z}(\varrho, 0+) &= \frac{12\mu h \varrho^2}{R^5}, \\ \sigma^0_{\phi z}(\varrho, 0+) &= 0, \\ \sigma^0_{zz}(\varrho, 0+) &= -\frac{12\mu h^2 \varrho}{R^5}, \end{aligned} \tag{47}$$

where $R = (\varrho^2 + h^2)^{1/2}$.

When the pore is present we use the same velocity and stress expressions as in the previous problem, and the no-slip conditions on $z = 0+$, $\varrho > 1$, again lead to (28) and (29) (with a different value of C_1). The stress continuity conditions across the mouth of the pore are modified by using (47) instead of (27), but analogous calculations lead to the conditions

$$\begin{aligned} 2\frac{\partial \chi}{\partial z} - \frac{\partial \theta}{\partial z} &= C_2 \varrho + \frac{2\varrho h}{R^3}, \\ 2\frac{\partial \chi}{\partial z} + \frac{\partial \theta}{\partial z} &= \frac{1}{\varrho}\left\{\frac{6h}{R} - \frac{2h^3}{R^3} + C_3\right\} \end{aligned} \tag{48}$$

on $z = 0+, 0 \leqslant \varrho < 1$, where C_2, C_3 are constants of integration. As in (17), the boundedness of $(48)_2$ as $\varrho \to 0$ requires that $C_3 = -4$, and on $z = 0+$, $0 \leqslant \varrho < 1$ we have

$$\begin{aligned} \frac{\partial \chi}{\partial z} &= \tfrac{1}{4}C_2\varrho - \frac{1}{\varrho} + \frac{1}{\varrho R} + \frac{\varrho h}{R^3}, \\ \frac{\partial \theta}{\partial z} &= -\tfrac{1}{2}C_2\varrho - \frac{2}{\varrho} + \frac{2h}{\varrho R}. \end{aligned} \tag{49}$$

Similarly, from $(47)_3$ continuity of normal stress requires that on $z = 0+, 0 \leqslant \varrho < 1$,

$$\frac{\partial \psi}{\partial z} = -\frac{3\varrho h^2}{R^5}. \tag{50}$$

Equations (28), (29), (49) and (50) now supply the mixed boundary conditions for the determination of χ, θ, ψ, and each potential problem is of the same structure as in the previous section.

Consider first (29) and (50); the representation

$$\psi(\varrho, z) = \frac{1}{2\varrho}\int_{-1}^{1} \frac{(z + it)j(t)}{\{\varrho^2 + (z + it)^2\}^{1/2}}\,\mathrm{d}t, \tag{51}$$

where $j(t) = j(-t)$ and $\int_0^1 j(t)\,\mathrm{d}t = 0$, satisfies (29) identically and (5) requires that

$$\frac{\partial}{\partial \varrho} \int_0^{\varrho} \frac{j(t)}{(\varrho^2 - t^2)^{1/2}}\,\mathrm{d}t = \frac{3\varrho h^2}{R^5}, \quad 0 \leqslant \varrho < 1. \tag{52}$$

The solution of this equation subject to the integral constraint is found as

$$j(t) = \frac{2}{\pi h}\left\{\frac{t^2(t^2 + 3h^2)}{(t^2 + h^2)^2} - \frac{1}{h^2 + 1}\right\}, \quad 0 \leqslant t \leqslant 1. \tag{53}$$

In order to determine χ we proceed as in the shear-flow problem using the decomposition provided by (36) and (37). A lengthy calculation, similar to that leading to (40) but using $(49)_1$ in place of $(32)_1$, shows that now the relevant function $g(t)$ in (37) is

$$\begin{aligned} g(t) = {} & \frac{2}{\pi} + \frac{C_2}{2\pi}\left(\tfrac{1}{3} - t^2\right) + \frac{1}{\pi}\log\left(\frac{h^2 + t^2}{h^2 + 1}\right) \\ & + \frac{2}{\pi}\left\{\frac{h^2}{t^2 + h^2} - 2h\tan^{-1}\left(\frac{1}{h}\right)\right\}, \quad 0 \leqslant t \leqslant 1. \end{aligned} \tag{54}$$

Further, employing the decomposition (41) for θ and condition $(49)_2$, the function $h(t)$ appropriate to the present problem is

$$h(t) = \frac{4}{\pi} + \frac{C_2}{\pi}\left(t^2 - \tfrac{1}{3}\right) + \frac{2}{\pi}\log\left(\frac{h^2 + t^2}{h^2 + 1}\right) - \frac{4h}{\pi}\tan^{-1}\left(\frac{1}{h}\right), \quad 0 \leqslant t \leqslant 1. \tag{55}$$

The two remaining constants C_1 and C_2 are computed by invoking the minimum stress-singularity conditions (43), whence

$$C_1 = \frac{8}{\pi}\left\{1 - h\tan^{-1}\left(\frac{1}{h}\right) - \frac{1}{3(h^2 + 1)}\right\}, \tag{56}$$

$$C_2 = -\frac{2}{h^2 + 1}. \tag{57}$$

The functions χ, θ, ψ are now completely determined but unfortunately the somewhat complicated nature of $j(t)$, $g(t)$ and $h(t)$ does not permit complete evaluation of the contour integrals to give simple closed forms comparable with (45) and (46).

As an application of this solution we calculate an approximation to the drag experienced by a small particle when incident with the z-axis and moving with a velocity $U\mathbf{i}$, where $\mathbf{i}$ is a unit vector parallel to the x-axis. It is assumed that the particle translates without rotation parallel to a principal axis of resistance, and we denote by $-F\mathbf{i}$ and $-F_\infty\mathbf{i}$ the viscous drag forces on the particle in the presence of the membrane and in an everywhere unbounded viscous fluid. Let c be the pore radius in physical units and set $b = ch$. Then, if a denotes a typical dimension of the translating body, according to Brenner [3]

$$\frac{F}{F_\infty} = \frac{1}{1 - k_1(F_\infty/8\pi\mu Ub)} + \mathrm{O}(a/l)^3, \tag{58}$$

where $l = \max(b, c)$, and the drag factor k_1 is defined by

$$k_1 = -\mathbf{v}^*(Q) \cdot \mathbf{i}. \tag{59}$$

In (59) $\mathbf{v}^*(Q)$ is the regular part of the velocity field in $z > 0$, that is $\mathbf{v}^{\mathrm{I}}(\mathbf{r}) - \mathbf{v}_\infty(\mathbf{r}, h)$, evaluated at the centre $Q(0, 0, h)$ of the body. Thus,

$$k_1 = -u^*(\varrho = 0, z = h), \tag{60}$$

where

$$u^* = \frac{2}{R_2} - \frac{(z+h)^2}{R_2^3} + z\frac{\partial^2 \chi_0}{\partial\varrho\partial z} + \frac{\partial \chi_0}{\partial \varrho} + \frac{\theta_0}{\varrho} + z\frac{\partial^2 \chi}{\partial\varrho\partial z} + \frac{\partial \chi}{\partial \varrho} + \frac{\theta}{\varrho} + z\frac{\partial \psi}{\partial \varrho}. \tag{61}$$

In (61), χ_0 and θ_0 are given by (19), (20), with χ, θ, ψ as detailed in the previous paragraphs.

To evaluate (60) from (61) we begin by noting that

$$z\frac{\partial^2 \chi_0}{\partial\varrho\partial z} + \frac{\partial \chi_0}{\partial \varrho} + \frac{\theta_0}{\varrho} = -\frac{5}{4h} \tag{62}$$

when $\varrho = 0$, $z = h$. We can also show that when $z = h$,

$$\lim_{\varrho\to 0}\frac{\theta}{\varrho} = \frac{1}{2}\int_0^1 \frac{th'(t)}{h^2 + t^2}\,\mathrm{d}t,$$

$$\lim_{\varrho\to 0}\frac{\partial \chi}{\partial \varrho} = \frac{1}{2}\int_0^1 \frac{tg'(t)}{h^2 + t^2}\,\mathrm{d}t,$$

$$\lim_{\varrho\to 0}\frac{\partial^2 \chi}{\partial\varrho\partial z} = -h\int_0^1 \frac{tg'(t)}{(h^2 + t^2)^2}\,\mathrm{d}t,$$

$$\lim_{\varrho\to 0}\frac{\partial \psi}{\partial \varrho} = -\frac{1}{2}\int_0^1 \frac{(h^2 - t^2)}{(h^2 + t^2)}\,j(t)\,\mathrm{d}t.$$

Inserting these results in (61) then gives

$$u^*(0, h) = -\frac{3}{4h} + \frac{1}{2}\int_0^1 \frac{th'(t)}{h^2 + t^2}\,\mathrm{d}t - \frac{1}{2}\int_0^1 \frac{(h^2 - t^2)\{tg'(t) + hj(t)\}}{(h^2 + t^2)^2}\,\mathrm{d}t, \tag{63}$$

which on substituting (53) through (57) and integrating supplies the value

$$k_1 = \frac{5 + 3h^2}{2\pi(1 + h^2)^2} + \frac{3}{2\pi h} \tan^{-1} h. \tag{64}$$

We note two limiting cases of (64); in the limit $h \to 0$, i.e., the particle is in the mouth of the pore, we have

$$k_1 = \frac{4}{\pi}, \tag{65}$$

whereas for large h,

$$k_1 \sim \frac{3}{4h}, \tag{66}$$

the value appropriate to a Stokeslet placed parallel to and distance h from a rigid unperforated plate. For intermediate values of h, k_1 decreases monotonically from the value (65) to zero as $h \to \infty$. By combining (65) with the drag factor k_2 given in [1] for the axisymmetric case, namely

$$k_2 = \frac{1}{\pi}\left\{\frac{3 \tan^{-1} h}{h} + \frac{3}{1 + h^2} - \frac{4}{(1 + h^2)^2}\right\}, \tag{67}$$

the drag force on a body crossing the pore axis at an arbitrary angle can be computed.

The approach of this section is easily applied to cases in which the Stokeslet is replaced by other singularities such as rotlets, potential dipoles and stresslets, and other shear profiles can be considered. The methods also form an essential ingredient in extending the calculations to encompass circular cylindrical pores of infinite or finite lengths when the flow field is asymmetric. Results pertaining to these configurations will be reported in future papers.

4. Shear flow past an elliptic pore

Suppose that a shear flow parallel to the x-axis and given by (26) exists in $z > 0$, and that the circular pore in $z = 0$ is replaced by one whose boundary is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let S be the interior of this ellipse, and $\bar{S}$ the complement of S in $z = 0$. An appropriate representation of the velocity fields in $z > 0$ and $z < 0$ in terms of two harmonic functions $f(x, y, z)$ and $g(x, y, z)$, suggested by work in elasticity [10], is

$$\mathbf{v}^{\mathrm{I}}(x, y, z) = Uz\mathbf{i} + \mathbf{V}(x, y, z), \quad \mathbf{v}^{\mathrm{II}}(x, y, z) = \mathbf{V}(x, y, z), \tag{68}$$

where the Cartesian components of **V** are

$$\begin{aligned} V_x &= -\frac{\partial f}{\partial z} + z\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x \partial y}\right), \\ V_y &= -\frac{\partial g}{\partial z} + z\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 g}{\partial y^2}\right), \\ V_z &= z\left(\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 g}{\partial y \partial z}\right), \end{aligned} \tag{69}$$

the associated pressure field being

$$p = 2\mu\left(\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial y \partial z}\right). \tag{70}$$

In (69), (70) the harmonic functions f and g are odd functions of z, and the relevant Cartesian stress components in $z > 0$ follow as

$$\begin{aligned} \sigma^1_{xz} &= \mu\left(U - \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x \partial y} + 2z\frac{\partial^2 F}{\partial x \partial z}\right), \\ \sigma^1_{yz} &= \mu\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 g}{\partial z^2} + 2z\frac{\partial^2 F}{\partial y \partial z}\right), \\ \sigma^1_{zz} &= 2\mu z\frac{\partial^2 F}{\partial z^2} \end{aligned} \tag{71}$$

where

$$F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y},$$

those in $z < 0$ resulting from setting $U = 0$ in (71).

From (69) the no-slip boundary conditions on $\bar{S}$ require that

$$\frac{\partial f}{\partial z} = 0, \quad \frac{\partial g}{\partial z} = 0 \quad \text{on } z = 0+, \quad (x, y) \in \bar{S}, \tag{72}$$

whilst from (71) continuity of σ_{xz} and σ_{yz} across S provide the conditions

$$\left.\begin{aligned} -\frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x \partial y} &= -\tfrac{1}{2}U, \\ -\frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial x \partial y} &= 0 \end{aligned}\right\} \quad \text{on } z = 0+, \quad (x, y) \in S. \tag{73}$$

Further f and g must tend to zero as $x^2 + y^2 + z^2 \to \infty$.

In order to determine f and g from (72), (73) we introduce ellipsoidal coordinates (α, β, γ), related to Cartesian coordinates by the transformation

$$
\begin{aligned}
x &= ak^2 \,\mathrm{sn}\alpha \,\mathrm{sn}\beta \,\mathrm{sn}\gamma, \\
y &= -ak^2 k'^{-1} \,\mathrm{cn}\alpha \,\mathrm{cn}\beta \,\mathrm{cn}\gamma, \\
z &= iak'^{-1} \,\mathrm{dn}\alpha \,\mathrm{dn}\beta \,\mathrm{dn}\gamma,
\end{aligned} \tag{74}
$$

where k, the eccentricity of the ellipse $x^2/a^2 + y^2/b^2 = 1$, is the modulus of the Jacobian elliptic functions in (74) and $k' = (1 - k^2)^{1/2}$ is the complementary modulus. To obtain all values of x, y and z, it is necessary for α, β and γ to vary in the ranges α from $-2K$ to $2K$, β from K to $K + 2iK'$, γ from iK' to $K + iK'$, where $K(k)$ and $K' = K(k')$ are the usual complete elliptic integrals of the first kind. The coordinate surfaces $\alpha =$ constant and $\beta =$ constant are hyperboloids of two sheets and one sheet, respectively, whilst $\gamma =$ constant is an ellipsoid. In particular $\gamma = K + iK'$ gives S and $\beta = K + iK'$ corresponds to $\bar{S}$.

Consider the region $z \geqslant 0$; since f is harmonic so is $\partial f/\partial z$, and using ellipsoidal harmonic functions given in [11, 12] we express the derivative as

$$
\frac{\partial f}{\partial z} = A \,\mathrm{dn}\alpha \,\mathrm{dn}\beta \,\frac{dF_1^0(\gamma)}{dF_1^0\,(K + iK')}, \tag{75}
$$

where A is a constant and $dF_1^0(\gamma)$ a Lamé function of the second kind given by

$$
dF_1^0(\gamma) = -\frac{3k^3}{k'^2} \,\mathrm{dn}\gamma \left\{ E(\gamma) - \frac{k^2 \,\mathrm{sn}\gamma \,\mathrm{cn}\gamma}{\mathrm{dn}\gamma} - i(K' - E') \right\}, \tag{76}
$$

with $E(\gamma) = \int_0^\gamma \mathrm{dn}^2 u \, du$, an elliptic integral of the second kind. Similarly, writing

$$
\frac{\partial g}{\partial z} = B \,\mathrm{dn}\alpha \,\mathrm{dn}\beta \,\frac{dF_1^0(\gamma)}{dF_1^0(K + iK')} \tag{77}
$$

for some constant B, (72) is satisfied by virtue of the fact that $\mathrm{dn}\,(K + iK') = 0$.

In order to apply (73) the various second-order derivatives of f and g must be evaluated on S. We first note that on S,

$$
\frac{\partial f}{\partial z} = A \,\mathrm{dn}\alpha \,\mathrm{dn}\beta = Ak' \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2},
$$

whence

$$
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = -\frac{Ak'^2 k \,\mathrm{sn}\alpha \,\mathrm{sn}\beta}{\mathrm{dn}\alpha \,\mathrm{dn}\beta}. \tag{78}
$$

Now, on S, $\partial/\partial z = -\partial/a \operatorname{dn}\alpha \operatorname{dn}\beta\, \partial\gamma$, and (78) can be rewritten as

$$\frac{\partial}{\partial\gamma}\left(\frac{\partial f}{\partial x}\right) = Ak'^2 k \operatorname{sn}\alpha \operatorname{sn}\beta \text{ on } S.$$

Thus,

$$\frac{\partial f}{\partial x} = Ak'^2 k \operatorname{sn}\alpha \operatorname{sn}\beta \frac{sF_1^0(\gamma)}{sF_1^{0\prime}(K + iK')}, \tag{79}$$

where $sF_1^0(\gamma)$ is a further Lamé function of the second kind given by

$$sF_1^0(\gamma) = 3k \operatorname{sn}\gamma \{\gamma - iK' - E(\gamma - iK')\}. \tag{80}$$

In particular on S, i.e, $\gamma = K + iK'$, (78) shows that

$$\frac{\partial f}{\partial x} = \frac{Ak'^2(K - E)}{ak^2} x, \tag{81}$$

where $E(k)$ is the complete elliptic integral of the second kind. It now follows from (81) that on S

$$\frac{\partial^2 f}{\partial x^2} = \frac{Ak'^2(K - E)}{ak^2}, \qquad \frac{\partial^2 f}{\partial x \partial y} = 0. \tag{82}$$

The remaining derivative $\partial^2 f/\partial z^2$ on S is obtained directly from (75) as

$$\frac{\partial^2 f}{\partial z^2} = -\frac{AE}{a}, \tag{83}$$

using the previously quoted expression for $\partial/\partial z$.

It is immediately apparent from $(82)_2$ and $(73)_2$ that $B = 0$ and $g \equiv 0$, whilst (82), (83) and $(73)_1$ require that

$$\frac{AE}{a} + \frac{Ak'^2(K - E)}{ak^2} = -\frac{U}{2}, \tag{84}$$

that is

$$A = -\frac{aUk^2}{2\{(k^2 - k'^2)E + k'^2 K\}}, \tag{85}$$

and the solution is complete.

As an example of a quantity of physical interest, consider the components of velocity in the pore mouth $\gamma = K + iK'$. We have that when $\gamma = K + iK'$,

$$v_x^{\mathrm{I}} = -\frac{\partial f}{\partial z} = \frac{aUk^2}{2\{(k^2 - k'^2)E + k'^2K\}} \,\mathrm{dn}\alpha\, \mathrm{dn}\beta$$

$$= \frac{aUk^2k'}{2\{(k^2 - k'^2)E + k'^2K\}}\left\{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right\}^{1/2},$$

and

$$v_y^{\mathrm{I}} = 0. \tag{87}$$

In the limit $k \to 0$,

$$v_x^{\mathrm{I}} \to \frac{2aU}{3\pi}\left(1 - \frac{\varrho^2}{a^2}\right)^{1/2}, \tag{88}$$

a result for a circular pore of radius a which can be verified using the solution of Section 3(a). Equations (86), (87) indicate that in the pore mouth the fluid velocity is parallel to the shear and is constant on ellipses concentric with and similar to the boundary of the pore. Note also that the appropriate slit solution of Smith [5] can be recovered by a suitable limiting process applied to the semi-axes of the ellipse.

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